

On a compressible non-isothermal model for nematic liquid crystals

Boling Guo¹, Binqiang Xie^{2*}, Xiaoyu Xi³

¹*Institute of Applied Physics and Computational Mathematics, China Academy of Engineering Physics, Beijing, 100088, P. R. China*

²*Graduate School of China Academy of Engineering Physics, Beijing, 100088, P. R. China*

³*Graduate School of China Academy of Engineering Physics, Beijing, 100088, P. R. China*

Abstract

We prove the existence of a weak solution to a non-isothermal compressible model for nematic liquid crystals. An initial-boundary value problem is studied in a bounded domain with large data. The existence of a global weak solution is established through a three-level approximation, energy estimates, and weak convergence for the adiabatic exponent $\gamma > \frac{3}{2}$.

Keywords: weak solutions; compressible non-isothermal model; nematic liquid crystals.

2010 Mathematics Subject Classification: 76W05, 35Q35, 35D05, 76X05.

1 Introduction

The evolution of liquid crystals in $\Omega \subset R^3$ is described by the following system

$$\partial_t(\rho) + \operatorname{div}(\rho u) = 0, \quad (1.1a)$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho, \theta) = \operatorname{div} \mathbb{S} - \nu \operatorname{div}(\nabla d \odot \nabla d - (\frac{1}{2}|\nabla d|^2 + F(d))\mathbb{I}), \quad (1.1b)$$

$$\partial_t(\rho \theta) + \operatorname{div}(\rho \theta u) + \operatorname{div} q = \mathbb{S} : \nabla u - R \rho \theta \operatorname{div} u + |\Delta d - f(d)|^2, \quad (1.1c)$$

$$\partial_t d + u \cdot \nabla d = \kappa(\Delta d - f(d)), \quad |d| = 1, \quad (1.1d)$$

where the functions ρ, u, θ and d represent the mass density, the velocity field, the absolute temperature and the unit vector field that represents the macroscopic molecular orientation of the liquid crystal material. P stands for the pressure, \mathbb{S} denotes the viscous stress tensor. The positive constants ν, κ denote the competition between kinetic energy and potential energy, and microscopic elastic relation time for the molecular orientation field, respectively. $\nabla d \odot \nabla d$ denotes the 3×3 matrix whose ij th entry is $\langle \partial_{x_i} d, \partial_{x_j} d \rangle$. The vector-valued smooth function $f(d)$ denotes the penalty function and has the following form:

$$f(d) = \nabla_d F(d), \quad (1.2)$$

Email: gbl@iapcm.ac.cn(B.L.Guo), xbq211@163.com(B.Q.Xie), xixiaoyu1357@126.com(X.Y.Xi).

where the scalar function $F(d)$ is the bulk part of the elastic energy. A typical example is choose $F(d)$ as the Ginzburg-Landau penalization thus yielding the penalty function $f(d)$ as:

$$F(d) = \frac{1}{4\sigma_0^2}(|d|^2 - 1)^2, f(d) = \frac{1}{2\sigma_0^2}(|d|^2 - 1)d,$$

where $\sigma_0 > 0$ is a constant.

Our analysis is based on the following physically grounded assumptions:

[A1]The viscous stress tensor \mathbb{S} is determined by the Newton's rheological law

$$\mathbb{S} = \mu(\nabla_x u + \nabla_x^\perp u) + \lambda \operatorname{div}_x u \mathbb{I}, \quad (1.3)$$

where μ and λ are respectively the shear and bulk constant viscosity coefficients satisfying

$$\mu > 0, \lambda + \frac{2}{3}\mu \geq 0. \quad (1.4)$$

[A2]The internal pressure obeys the following equation of state:

$$P(\rho, \theta) = \rho^\gamma + R\rho\theta, \quad (1.5)$$

where R is the perfect gas constant. The first term describes the elastic pressure while the latter term represents the thermodynamic pressure for ideal gas given by Boyle's law.

[A3]The internal energy flux q is expressed through Fourier's law:

$$q = -\kappa(\theta)\nabla\theta, \kappa \geq 0, \quad (1.6)$$

A key element of the system (1.1) is the heat conducting coefficient κ which is a continuous function of temperature and satisfies the following growth condition:

$$\underline{\kappa}_0(1 + \theta^\alpha) \leq \kappa(\rho, \theta) \leq \overline{\kappa}_0(1 + \theta^\alpha), \quad (1.7)$$

where $\underline{\kappa}_0, \overline{\kappa}_0, \alpha$ are positive constants and $\alpha \geq 2$.

To complete the system (1.1), the boundary conditions are given by

$$u|_{\partial\Omega} = \nabla\theta \cdot n|_{\partial\Omega} = \nabla d \cdot n|_{\partial\Omega} = 0. \quad (1.8)$$

Of course, we also need to assume the initial conditions

$$\rho(0, \cdot) = \rho_0, (\rho u)(0, \cdot) = m_0, \theta(0, \cdot) = \theta_0, d(0, \cdot) = d_0. \quad (1.9)$$

together with the compatibility condition:

$$m_0 = 0 \quad \text{on the set} \quad \{x \in \Omega | \rho_0(x) = 0\}. \quad (1.10)$$

The purpose of this paper is to establish the global existence of weak solutions to this initial boundary value problem with large initial data. To our best knowledge, the only available existence results for non-isothermal model concern the incompressible case(for instance Feireisl-Rocca-Schimperna [2],[3] and the references therein), they proved global existence of weak solutions to the non-isothermal system with penalty term $f(d)$. Compared with the system with the penalty term $f(d)$, system with term $|\nabla d|^2 d$ was studied by J.K.Li and Z.P.Xin [5]. The isothermal case

with the incompressible condition was proposed by Lin in [8] and later analyzed by Lin-Liu in [9]. The model proposed in [9],[8] is a considerably simplified version of the famous Leslie-Ericksen model introduced by Ericksen [6] and Leslie [7] in the 1960's. But the regularity and uniqueness of weak solutions is still open, at least for three dimensional case, see Li-Liu [10] for the regularity results. When the density of the liquid crystals is taken account, the global existence of weak solutions can be obtained in the framework of [12],[11],[4], see Jiang-Tan [13] and Liu-Zhang [14] for the incompressible model, see Wang-Yu [1] for the compressible model. Therefore, we want to prove the global existence of weak solution to a simplified non-isothermal compressible model for nematic liquid crystals.

Now, we give the definition of a varational solution to (1.1)-(1.10).

Definition 1.1. *We call (ρ, u, θ, d) is as a varational weak solution to the problem (1.1)-(1.10), if the following is satisfied.*

(1) *the density ρ is a non-negative function satisfying the internal identity*

$$\int_0^T \int_{\Omega} \rho \partial_t \phi + \rho u \cdot \nabla \phi dx dt + \int_{\Omega} \rho_0 \phi(0) dx = 0, \quad (1.11)$$

for any test function $\phi \in C^\infty([0, T] \times \overline{\Omega})$, $\phi(T) = 0$. In addition, we require that ρ is a renormalized solution of the continuity equation (1.1a) in the sense that

$$\partial_t b(\rho) + \operatorname{div}[b(\rho)u] + [b'(\rho)\rho - b(\rho)]\operatorname{div}u = 0. \text{ in } D'(\Omega) \quad (1.12)$$

for any function $b \in C^1[0, \infty)$, such that $b'(z) = 0$ when z is big enough.

(2) *the velocity u belongs to the space $L^2(0, T; W_0^{1,2}(\Omega))$, the momentum equation in (1.1b) holds in $D'((0, T) \times \Omega)$ (in the sense of distributions), that means,*

$$\begin{aligned} & \int_{\Omega} m_0 \phi(0) dx + \int_0^T \int_{\Omega} \rho u \cdot \partial_t \phi + \rho(u \otimes u) : \nabla \phi + P \operatorname{div} \phi dx dt \\ &= \int_0^T \int_{\Omega} [\mathbb{S} - \nu \nabla d \odot \nabla d - \nu(\frac{1}{2}|\nabla d|^2 + F(d))\mathbb{I}] : \nabla \phi dx dt, \end{aligned} \quad (1.13)$$

for any test function $\phi \in C^\infty([0, T] \times \overline{\Omega})$, $\phi(T) = 0$.

(3) *the temperature θ is a non-negative function satisfying*

$$\begin{aligned} & \int_{\Omega} \rho_0 \theta_0 \phi(0) dx + \int_0^T \int_{\Omega} \rho \theta \cdot \partial_t \phi + \rho \theta u \cdot \nabla \phi + \mathcal{K}(\theta) \Delta \phi dx dt \\ & \leq \int_0^T \int_{\Omega} (R \rho \theta \operatorname{div} u - \mathbb{S} : \nabla u - |\Delta d - f(d)|^2) \phi dx dt, \end{aligned} \quad (1.14)$$

for any test function $\phi \in C^\infty([0, T] \times \Omega)$,

$$\phi \geq 0, \quad \phi(T) = 0, \quad \nabla \phi \cdot n|_{\partial\Omega} = 0; \quad (1.15)$$

Here $\mathcal{K}(\theta) = \int_0^\theta k(z) dz$.

(4) *The equation of director field d hold in $D'((0, T) \times \Omega)$ in the sense that*

$$\int_{\Omega} d_0 \phi(0) dx + \int_0^T \int_{\Omega} d \cdot \partial_t \phi - u \cdot \nabla d \phi dx dt = \theta \int_0^T \int_{\Omega} -d \Delta \phi + f(d) \phi dx dt, \quad (1.16)$$

for any test function $\phi \in C^\infty([0, T] \times \Omega)$,

$$\phi \geq 0, \quad \phi(T) = 0, \quad \nabla \phi \cdot n|_{\partial\Omega} = 0; \quad (1.17)$$

(5) The energy inequality

$$E(\rho, u, \theta, d)(\tau) = \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{|\nabla d|^2}{2} + F(d) + \rho \theta \right) dx \leq E(\rho, u, \theta, d)(0). \quad (1.18)$$

holds for a.a. $\tau \in (0, T)$, with

$$E(\rho, u\theta, d)(0) = \int_{\Omega} \frac{1}{2} \frac{|m_0|^2}{\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} + \frac{|\nabla d_0|^2}{2} + F(d_0) + \rho_0 \theta_0 dx; \quad (1.19)$$

(6) The function $\rho, \rho u, \rho \theta, d$ satisfy the initial conditions (1.9) in the weak sense,

$$\begin{cases} \operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega} \rho(t) \eta dx = \int_{\Omega} \rho_0 \eta dx, \\ \operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega} (\rho u)(t) \cdot \eta dx = \int_{\Omega} m_0 \cdot \eta dx, \\ \operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega} (\rho \theta)(t) \eta dx = \int_{\Omega} \rho_0 \theta_0 \eta dx, \\ \operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega} d(t) \cdot \eta dx = \int_{\Omega} d_0 \eta dx, \end{cases}$$

for all test function $\eta \in \mathcal{D}(\Omega)$.

Now, we are ready to formulate the main result of this paper.

Theorem 1.2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$ and $\gamma > \frac{3}{2}$. Assume that the pressure p , the conductivity coefficient and the viscosity coefficient satisfy the condition (1.4)-(1.7). Let the initial data satisfy

$$\begin{cases} \rho_0 \geq 0, \rho_0 \in L^\gamma(\Omega), \\ \frac{|m|^2}{\rho_0} \in L^1(\Omega), \\ \theta_0 \in L^\infty(\Omega), \theta_0 \geq C > 0 \text{ a.e. in } \Omega, \\ d_0 \in H^1(\Omega), F(d_0) \in L^1(\Omega), \end{cases} \quad (1.20)$$

If there exists a constant $C_0 > 0$, such that $d \cdot f(d) \geq 0$ for all $|d| \geq C_0 > 0$. Then problem (1.1)-(1.10) posses at least one variational solution ρ, u, θ, d on the interval such that

$$\rho \in L^\infty(0, T; L^\gamma(\Omega)) \cap C([0, T]; L^1(\Omega)), \quad (1.21)$$

$$u \in L^2(0, T; W_0^{1,2}(\Omega)), \quad \rho u \in C([0, T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega)), \quad (1.22)$$

$$\theta \in L^{\alpha+1}((0, T) \times \Omega), \quad \rho \theta \in L^\infty(0, T; L^1(\Omega)), \quad (1.23)$$

$$d \in L^\infty(0, T; W^{1,2}(\Omega)), \quad d \in L^2(0, T; W^{2,2}(\Omega)), \quad (1.24)$$

This paper is organized as follows. In section 2, we deduce a priori estimates from (1.1). In section 3, we establish the global existence of solutions to the Faedo-Galerkin approximation to (1.1). In section 4 and 5, we use the uniform estimates to recover the original system by vanishing the artificial viscosity and artificial pressure respectively, therefore the main theorem is proved by using the weak convergence method in the framework of Firesel [4].

2 A priori bounds

In this section, we collect the available a priori estimates. For the sake of simplicity, we set $\nu = \kappa = 1$. Firstly we formally derive the energy equality and some a priori estimates, which will play a very important role in our paper. Multiplying the equation (1.1b) by u , integrating over Ω , and using the boundary condition, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma-1} \right) dx + \int_{\Omega} \mathbb{S} : \nabla u dx &= \int_{\Omega} R \rho \theta \operatorname{div} u dx \\ &- \int_{\Omega} \operatorname{div} (\nabla d \odot \nabla d - (\frac{1}{2} |\nabla d|^2 + F(d)) \mathbb{I}_3) u dx. \end{aligned} \quad (2.1)$$

Using the equality

$$\operatorname{div} (\nabla d \odot \nabla d) = \nabla \left(\frac{1}{2} |\nabla d|^2 \right) + (\nabla d)^T \cdot \Delta d,$$

We have

$$\begin{aligned} \int_{\Omega} \operatorname{div} (\nabla d \odot \nabla d - (\frac{1}{2} |\nabla d|^2 + F(d)) \mathbb{I}_3) u dx \\ = \int_{\Omega} (\nabla d)^T \cdot \Delta d \cdot u dx - \int_{\Omega} \nabla_d F(d) u dx. \end{aligned} \quad (2.2)$$

Hence, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma-1} \right) dx + \int_{\Omega} \mathbb{S} : \nabla u dx \\ = \int_{\Omega} R \rho \theta \operatorname{div} u dx - \int_{\Omega} (\nabla d)^T \cdot \Delta d \cdot u dx + \int_{\Omega} \nabla_d F(d) u dx. \end{aligned} \quad (2.3)$$

Multiplying by $\Delta d - f(d)$ on the both sides of the equation in (1.1d) and integrating over Ω , we get

$$\begin{aligned} - \frac{d}{dt} \int_{\Omega} \left(\frac{|\nabla d|^2}{2} + F(d) \right) dx - \int_{\Omega} \nabla_d F(d) u dx \\ + \int_{\Omega} (\nabla d)^T \cdot \Delta d \cdot u dx = \int_{\Omega} |\Delta d - f(d)|^2 dx. \end{aligned} \quad (2.4)$$

Then, adding (2.3) and (2.4), we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{|\nabla d|^2}{2} + F(d) \right) dx \\ + \int_{\Omega} (\mathbb{S} : \nabla u + |\Delta d - f(d)|^2) dx = \int_{\Omega} R \rho \theta \operatorname{div} u dx. \end{aligned} \quad (2.5)$$

Integrating the equation (1.1c) over Ω and summing with (2.5), we have

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{|\nabla d|^2}{2} + F(d) + \rho \theta \right) dx = 0. \quad (2.6)$$

thus we obtain the total energy conservation. Assume the initial total energy is finite, we immediately obtain the following bounds:

$$\|\sqrt{\rho} u\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C, \quad (2.7)$$

$$\|d\|_{L^\infty(0,T;W^{1,2}(\Omega))} \leq C, \|F(d)\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad (2.8)$$

$$\|\rho\theta\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad (2.9)$$

Next, we will deduce the entropy estimates. We replaced the thermal energy equation (1.1c) with the entropy equation:

$$\theta(\partial_t(\rho s) + \operatorname{div}(\rho s u)) = \operatorname{div}(\kappa(\theta)\nabla\theta) + \mathbb{S} : \nabla u + |\Delta d - f(d)|^2, \quad (2.10)$$

where

$$s = \log \theta - \log \rho, \quad (2.11)$$

The entropy equation integrating over Ω gives to the integral identity

$$\begin{aligned} & \int_0^\tau \int_\Omega \frac{\kappa(\theta)|\nabla\theta|^2}{\theta^2} + \frac{\mathbb{S} : \nabla u}{\theta} + \frac{|\Delta d - f(d)|^2}{\theta} dxdt \\ &= \int_\Omega (\rho s)(\tau) dx - \int_\Omega (\rho s)_0 dx, \text{ for any } \tau \in [0, T], \end{aligned} \quad (2.12)$$

It is not hard to see that the density dependent part of the entropy is dominated by the elastic part of the internal energy:

$$|\rho \log \rho| \leq C(1 + \rho^\gamma) \leq C, \text{ for a certain } C > 0. \quad (2.13)$$

Moreover, we have

$$|\rho \log \theta| \leq \rho \theta \leq C, \quad (2.14)$$

Consequently, relation (2.9) entails

$$\begin{aligned} & \int_0^T \int_\Omega \frac{\kappa(\theta)|\nabla\theta|^2}{\theta^2} + \frac{\mathbb{S} : \nabla u}{\theta} + \frac{|\Delta d - f(d)|^2}{\theta} dxdt - \operatorname{ess\,inf}_{t \in [0, T]} \int_\Omega \rho \log(\theta) dx \\ & \leq C - \int_\Omega (\rho s)_0 dx, \end{aligned} \quad (2.15)$$

Now we can use (2.12) together with hypothesis (1.6) to discover the estimates

$$\int_0^T \int_\Omega |\nabla \theta^{\frac{\alpha}{2}}|^2 + |\nabla \log \theta|^2 + \frac{|\nabla u|^2}{\theta} + \frac{|\Delta d - f(d)|^2}{\theta} dxdt + \sup_{t \in [0, T]} \int_\Omega \rho |\log(\theta)| dx \leq C, \quad (2.16)$$

At this stage we shall need the following auxiliary result.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Assume that r is a non-negative function such that*

$$0 < M_0 \leq \int_\Omega r dx, \int_\Omega r^\gamma dx \leq K, \text{ for a certain } \gamma > 1 \quad (2.17)$$

Then

$$\|\xi\|_{W^{1,p}(\Omega)} \leq C(p, M_0, K) \|\nabla \xi\|_{L^p(\Omega)} + \int_\Omega r |\xi| dx \quad (2.18)$$

For the proof of this lemma, we refer to book [4].

Combining the conclusion of Lemma 2.1 and the estimates (2.13) we get

$$\theta^{\frac{\alpha}{2}} \text{ bounded in } L^2(0, T; W^{1,2}(\Omega)), \quad (2.19)$$

and

$$\log \theta \text{ bounded in } L^2((0, T) \times \Omega), \quad (2.20)$$

Now, we can integrate the forth equation in (1.1) to obtain

$$\int_0^T \int_{\Omega} \mathbb{S} : \nabla u + |\Delta d - f(d)|^2 dx dt \leq \int_0^T \int_{\Omega} R \rho \theta |\operatorname{div} u| dx dt + C, \quad (2.21)$$

Seeing that, by virtue of Holder's inequality, we have

$$\|R \rho \theta\|_{L^2((0, T) \times \Omega)} \leq C \|\theta\|_{L^2([0, T]; L^{\frac{2\gamma}{\gamma-2}}(\Omega))} \|\rho\|_{L^\infty([0, T]; L^\gamma(\Omega))}, \quad (2.22)$$

One can use the estimates (2.16) together with the Sobolev imbedding theorem to conclude that

$$\int_0^T \int_{\Omega} |R \rho \theta|^2 dx dt \leq C, \quad (2.23)$$

Here we require $\alpha > \frac{2\gamma}{3(\gamma-2)}$.

In accordance with hypothesis (1.3), the relation (2.18), (2.20) give rise to the estimate

$$\|\nabla u\|_{L^2((0, T) \times \Omega)} \leq C, \quad (2.24)$$

and

$$\|\Delta d - f(d)\|_{L^2((0, T) \times \Omega)} \leq C, \quad (2.25)$$

To control the strongly nonlinear terms containing ∇d , we need more regularity for the direction field d . To deal with this obstacle, we have the following lemma:

Lemma 2.2. *If there exists a constant $C_0 > 0$ such that $d \cdot f(d) \geq 0$ for all $|d| \geq C_0 > 0$, then $d \in L^\infty((0, T) \times \Omega)$ and $\nabla d \in L^4((0, T) \times \Omega)$.*

For the proof of this lemma, we refer to paper [1].

As for regular solutions the temperature is always positive, we are allowed to multiply the thermal energy equation by $\theta^{-\omega}$, $0 < \omega \leq 1$. By parts integration yields

$$\begin{aligned} & \omega \int_0^\tau \int_{\Omega} \frac{\kappa(\theta) |\nabla \theta|^2}{\theta^{\omega+1}} + \frac{\mathbb{S} : \nabla u}{\theta^\omega} + \frac{|\Delta d - f(d)|^2}{\theta^\omega} dx dt \\ &= \left[\int_{\Omega} \rho \theta^{1-\omega} dx \right]_{t=0}^{t=\tau} + \theta^{1-\omega} R \rho \operatorname{div} u dx dt, \end{aligned} \quad (2.26)$$

whence, in accordance with hypothesis (1.7), we have

$$\int_0^\tau \int_{\Omega} |\nabla \theta^{\frac{\alpha+1-\omega}{2}}|^2 dx dt \leq C, \quad (2.27)$$

3 The approximation system

In this section we introduce a three level approximating scheme which involves a system of regularized equations. The approximation scheme reads:

$$\partial_t(\rho) + \operatorname{div}(\rho u) = \varepsilon \Delta \rho, \quad (3.1a)$$

$$\begin{aligned} & \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho, \theta) + \delta \nabla \rho^\beta + \varepsilon \nabla u \cdot \nabla \rho \\ & = \operatorname{div} \mathbb{S} - \nu \operatorname{div}(\nabla d \odot \nabla d - (\frac{1}{2} |\nabla d|^2 + F(d)) \mathbb{I}), \end{aligned} \quad (3.1b)$$

$$\partial_t((\delta + \rho)\theta) + \operatorname{div}(\rho \theta u) - \Delta \mathcal{K}(\theta) + \delta \theta^{\alpha+1} = (1 - \delta) \mathbb{S} : \nabla u - R \rho \theta \operatorname{div} u + |\Delta d - f(d)|^2, \quad (3.1c)$$

$$\partial_t d + u \cdot \nabla d = \Delta d - f(d), \quad |d| = 1, \quad (3.1d)$$

with boundary conditions

$$\nabla \rho \cdot n|_{\partial \Omega} = 0, \quad (3.2a)$$

$$u|_{\partial \Omega} = 0, \quad (3.2b)$$

$$\nabla \theta \cdot n|_{\partial \Omega} = 0, \quad (3.2c)$$

$$d|_{\partial \Omega} = d_0, \quad (3.2d)$$

together with initial data

$$\rho|_{t=0} = \rho_{0,\delta}(x), \quad (3.3a)$$

$$\rho u|_{t=0} = m_{0,\delta}(x), \quad (3.3b)$$

$$(\delta + \rho)\theta|_{t=0} = (\delta + \rho_{0,\delta}(x))\theta_{0,\delta}(x), \quad (3.3c)$$

$$d|_{t=0} = d_{0,\delta}(x), \quad (3.3d)$$

Here the initial data $\rho_{0,\delta} \in C^{2+\nu}(\overline{\Omega})$, $\nu > 0$, satisfies the following conditions:

$$0 < \delta \leq \rho_{0,\delta}(x) \leq \delta^{-\frac{1}{2\beta}}, \quad (3.4)$$

and

$$\rho_{0,\delta} \rightarrow \rho_0 \text{ in } L^\gamma(\Omega), \quad |\{\rho_{0,\delta} < \rho_0\}| \rightarrow 0 \text{ as } \delta \rightarrow 0, \quad (3.5)$$

Moreover, the initial momenta $m_{0,\delta}$ are defined as

$$m_{0,\delta}(x) = \begin{cases} m_0, & \text{if } \rho_{0,\delta}(x) \geq \rho_0(x), \\ 0, & \text{if } \rho_{0,\delta}(x) < \rho_0(x). \end{cases} \quad (3.6)$$

Further, the function $\theta_{0,\delta} \in C^{2+\nu}(\overline{\Omega})$ satisfy

$$0 < \underline{\theta} \leq \theta_{0,\delta} \leq \overline{\theta}, \text{ for all } x \in \Omega, \quad \delta > 0, \quad \nabla \theta_{0,\delta} \cdot n|_{\partial \Omega} = 0. \quad (3.7)$$

3.1. Faedo-Galerkin method

The initial boundary value (3.1)-(3.7) will be solved via a modified Faedo-Galerkin method. Firstly, we introduce the finite-dimensional space endowed with the L^2 Hilbert space structure:

$$X_n = \text{span}\{\eta_i\}_{i=1}^n, \quad n \in 1, 2, \dots, \quad (3.8)$$

where the linearly independent functions $\eta_i \in \mathcal{D}(\Omega)^3$, $i=1,2,\dots$, form a dense subset in $C_0^2(\overline{\Omega}, R^3)$. The approximation solution $u_n \in C([0, T]; X_n)$ satisfy a set of integral equation of the following form :

$$\begin{aligned} & \int_{\Omega} \rho u_n(\tau) \cdot \eta dx - \int_{\Omega} m_{0,\delta} \cdot \eta dx \\ &= \int_0^\tau \int_{\Omega} (\text{div} \mathbb{S}_n - \text{div}(\rho u_n \otimes u_n) - \nabla(P(\rho) + \delta \nabla \rho^\beta) - \varepsilon \nabla u_n \cdot \nabla \rho) \cdot \eta dx dt \\ & - \int_0^\tau \int_{\Omega} \nu \text{div}(\nabla d \odot \nabla d - (\frac{1}{2} |\nabla d|^2 + F(d)) \mathbb{I}) \cdot \eta dx dt, \end{aligned} \quad (3.9)$$

Then the density $\rho_n = \rho[u_n]$ is determined uniquely as the solution of the following Neumann initial-boundary value problem (3.1a), (3.2a), (3.3a). The detail of this proof can be seen in their book [4] Lemma 7.1 and 7.2. The director field $d_n = d[u_n]$ is the unique solution of (3.1d), (3.2d), (3.3d), the proof of this result can refer to Lemma 3.1 and 3.2 in [1]. At the same time, $\theta_n = \theta[\rho_n, u_n, d_n]$ is the unique solution of (3.1c), (3.2c), (3.3c), the proof of this result can refer to Lemma 7.3 and 7.4 in [4]. Furthermore, the problem (3.9) can be solved at least on a short time interval $(0, T_n)$ with $T_n \leq T$ by a standard fixed point theorem on the Banach space $C([0, T]; X_n)$. We refer to [4] for more details. Thus we obtain a local solution $(\rho_n, u_n, d_n, \theta_n)$ in time.

To obtain uniform bounds on u_n , we derive an energy inequality similar to (2.5) as follows. Taking $\eta = u_n(t, x)$ with fixed t in (3.1b) and repeating the procedure for a priori estimates in Section 2, we deduce a total energy inequality :

$$\begin{aligned} & \int_{\Omega} (\frac{1}{2} \rho_n |u_n|^2 + \frac{\rho_n^\gamma}{\gamma-1} + \frac{\delta}{\beta} \rho_n^\beta + \frac{|\nabla d_n|^2}{2} + F(d_n) + \rho_n \theta_n + \delta \theta_n)(\tau) dx \\ & + \delta \int_0^\tau \int_{\Omega} \text{div} \mathbb{S}_n : \nabla u_n + \theta_n^{\alpha+1} dx dt + \varepsilon \int_0^\tau \int_{\Omega} \gamma |\nabla \rho^{\frac{\gamma}{2}}|^2 + \delta \beta |\nabla \rho^{\frac{\beta}{2}}|^2 dx dt \\ & = \int_{\Omega} (\frac{1}{2} \rho_n |u_n|^2 + \frac{\rho_n^\gamma}{\gamma-1} + \frac{\delta}{\beta} \rho_n^\beta + \frac{|\nabla d_n|^2}{2} + F(d_n) + \rho_n \theta_n + \delta \theta_n)(0) dx. \end{aligned} \quad (3.10)$$

From (3.10) we deduce that

$$u_n \text{ is bounded in } L^2(0, T; W_0^{1,2}(\Omega)), \quad (3.11)$$

by a constant that is independent of n and $T(n) \leq T$. Since all norms are equivalent on X_n , this implies that the approximate velocity fields u_n are bounded in $L^1(0, T; W^{1,\infty}(\Omega))$, then we know that the density ρ_n is bounded both from below and from above by a constant independent of $T(n) \leq T$. Consequently, the functions u_n remain bounded in X_n for any t independently of $T(n) \leq T$. These uniform estimates can extend the local solution u_n to the whole time interval $[0, T]$. Thus, we obtain the function (ρ_n, d_n, θ_n) on the whole time interval $[0, T]$.

3.2. First level of approximate solutions

The next step is to pass to the limit as $n \rightarrow \infty$ in the sequence of approximate solutions $\{\rho_n, u_n, d_n, \theta_n\}$. In order to achieve this, additional estimates are needed.

To begin with, by following a similar argument to Ferreisl [4], we get

$$\rho_n \rightarrow \rho \text{ in } L^\beta((0, T) \times \Omega).$$

By the energy estimate (3.10), we have

$$u_n \rightarrow u \text{ weakly } L^2(0, T; W_0^{1,2}(\Omega)).$$

$$\rho_n u_n \rightarrow \rho u \star \text{-weakly in } L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)).$$

where ρ, u satisfy equation (3.1a) together with boundary conditions (3.2a) and the initial condition holds in the sense of distribution.

At this stage of approximation, the absolute temperature θ_n is strictly positive, and, consequently, equation (3.1d) can be rewritten as follows

$$\begin{aligned} & \partial_t((\delta + \rho_n)\mathcal{H}(\theta_n)) + \operatorname{div}(\rho_n \mathcal{H}(\theta_n) u_n) - \Delta \mathcal{K}_h(\theta) + \delta \theta_n^{\alpha+1} h(\theta_n) \\ &= (1 - \delta) h(\theta_n) \mathbb{S}_n : \nabla u_n - \kappa(\theta_n) h'(\theta_n) |\nabla \theta_n|^2 - R \theta_n h(\theta_n) \rho_n \operatorname{div} u_n \\ &+ h(\theta_n) |\Delta d_n - f(d_n)|^2 + \varepsilon \Delta \rho_n (\mathcal{H}(\theta_n) - \theta_n h(\theta_n)), \end{aligned} \quad (3.12)$$

where $\mathcal{Q}_h, \mathcal{K}_h$ are determined by

$$\mathcal{H}(\theta) \equiv \int_0^\theta h(z) dz, \quad \mathcal{K}_h(\theta) = \int_0^\theta \kappa(z) h(z) dz. \quad (3.13)$$

Integrating (3.12) over Ω yields

$$\begin{aligned} & \frac{d}{dt} \int_\Omega (\delta + \rho_n) \mathcal{H}(\theta_n) dx + \delta \int_\Omega \theta_n^{\alpha+1} h(\theta_n) dx \\ &= \int_\Omega (1 - \delta) h(\theta_n) \mathbb{S}_n : \nabla u_n - \kappa(\theta_n) h'(\theta_n) |\nabla \theta_n|^2 - R \theta_n h(\theta_n) \rho_n \operatorname{div} u_n dx \\ &+ \int_\Omega h(\theta_n) |\Delta d_n - f(d_n)|^2 + \varepsilon (\nabla \rho \cdot \nabla \theta) \theta_n h'(\theta) dx, \end{aligned} \quad (3.14)$$

In particular, the choice $h(\theta_n) = (1 + \theta_n)^{-1}$ leads to relations

$$- \int_\Omega \kappa(\theta_n) h'(\theta_n) |\nabla \theta_n|^2 dx \geq C \int_\Omega |\nabla \theta_n^{\alpha/2}|^2 dx \quad (3.15)$$

while

$$\varepsilon \left| \int_\Omega (\nabla \rho \cdot \nabla \theta) \theta_n h'(\theta) dx \right| \leq \varepsilon \|\nabla \rho_n\|_{L^2(\Omega)} \|\nabla \log \theta_n\|_{L^2(\Omega)}, \quad (3.16)$$

and

$$\left| \int_\Omega R \theta_n h(\theta_n) \rho_n \operatorname{div} u_n dx \right| \leq C \|\rho_n\|_{L^2(\Omega)} \|\operatorname{div} u_n\|_{L^2(\Omega)}, \quad (3.17)$$

where we have used hypothesis (1.7). It follows from the energy estimates (3.10) that the right-hand side of the last inequality is bounded in $L^1(0, T)$ by a constant that depends only on δ .

Consequently, (3.14) integrated with respect to t together with the energy estimates yields a bound

$$\|\nabla \theta_n^{\alpha/2}\|_{L^2((0,T)\times\Omega)} \leq C, \quad (3.18)$$

By virtue of (3.18), we have

$$\theta_n \rightarrow \theta \text{ in } L^2(0, T; W^{1,2}(\Omega)). \quad (3.19)$$

Moreover, relation (3.10),(3.19) yield

$$\rho_n \theta_n \rightarrow \rho \theta \text{ in } L^2(0, T; L^q(\Omega)). \quad (3.20)$$

where

$$q = \frac{6\gamma}{\gamma+6} > \frac{6}{5}, \text{ provided } \gamma > 3/2.$$

Next, we will use the following variant Lions-Aubin Lemma:

Lemma 3.1. *Let v^n be a sequence of functions such that $L^{q_1}(0, T; L^{q_2}(\Omega))$, where $1 \leq p_1, p_2 \leq \infty$ and*

$$\{v_n\}_{n=1}^\infty \text{ is bounded in } L^2(0, T; L^q(\Omega)) \cap L^\infty(0, T; L^1(\Omega)) \text{ with } q > 2n/(n+2), \quad (3.21)$$

Let us assume in addition that

$$\partial_t v_n \geq \chi_n \text{ in } D'((0, T) \times \Omega), \quad (3.22)$$

where

$$\chi_n \text{ are bounded in } L^1(0, T; W^{-m,r}(\Omega))$$

for a certain $m \geq 1, r > 1$. Then $\{v_n\}_{n=1}^\infty$ contains a subsequence such that

$$v_n \rightarrow v \text{ in } L^2(0, T; W^{-1,2}(\Omega)).$$

Since θ_n satisfy the renormalized thermal energy (3.12), we can apply Lemma 3.1 together (3.20) to obtain

$$(\rho_n + \delta)\theta_n \rightarrow (\rho + \delta)\theta \text{ in } L^2(0, T; W^{-1,2}(\Omega)). \quad (3.23)$$

Relation (3.19) and (3.23) imply

$$(\rho_n + \delta)\theta_n \theta_n \rightarrow (\rho + \delta)\theta \theta \text{ weakly in } L^1((0, T) \times \Omega). \quad (3.24)$$

On the other hand, utilizing (3.10), (3.19) again, we get

$$(\rho_\varepsilon + \delta)\theta_n \theta_n \rightarrow (\rho + \delta)\overline{\theta\theta} \text{ weakly in } L^1((0, T) \times \Omega). \quad (3.25)$$

which compared with (3.24), yields

$$\overline{\theta^2} = \theta^2. \quad (3.26)$$

Therefore it is easy to see that (3.26) implies strong convergence θ_n in $L^2([0, T]; L^2(\Omega))$.

Now we integrating the equation (3.1c) over space-time, get

$$\int_0^T \int_\Omega (1 - \delta) \mathbb{S}_n : \nabla u_n + |\Delta d_n - f(d_n)|^2 dx dt \leq \int_0^T \int_\Omega R \rho_n \theta |\operatorname{div} u_n| dx dt + C, \quad (3.27)$$

Due to the estimate (3.18), we know the right hand of above inequality are bounded. Thus we have

$$\|\nabla u_n\|_{L^2((0,T)\times\Omega)} \leq C, \quad (3.28)$$

and

$$\|\Delta d_n - f(d_n)\|_{L^2((0,T)\times\Omega)} \leq C, \quad (3.29)$$

Following the procedure in section 2, we conclude that

$$d_n \in L^\infty([0, T]; H^1(\Omega)) \cap L^2([0, T]; H^2(\Omega)). \quad (3.30)$$

This yields that

$$\Delta d_n - f(d_n) \rightarrow \Delta d - f(d) \text{ weakly in } L^2([0, T]; L^2(\Omega)). \quad (3.31)$$

and

$$d_n \rightarrow d \text{ weakly in } L^\infty([0, T]; H^1(\Omega)) \cap L^2([0, T]; H^2(\Omega)). \quad (3.32)$$

We now apply the Aubin-Lions lemma to obtain the convergence of d_n and ∇d_n . From Lemma 2.1 in [4], we have

$$d_n \in L^\infty((0, T) \times \Omega). \quad (3.33)$$

and

$$\nabla d_n \in L^4((0, T) \times \Omega). \quad (3.34)$$

Using the equation (3.1d), we have

$$\begin{aligned} \|\partial_t d_n\|_{L^2(\Omega)} &\leq C\|u_n \cdot \nabla d_n\|_{L^2(\Omega)} + C\|\Delta d_n - f(d_n)\|_{L^2(\Omega)} \\ &\leq C\|u_n\|_{L^4(\Omega)}^2 + C\|\nabla d_n\|_{L^4(\Omega)}^2 + C\|\Delta d_n - f(d_n)\|_{L^2(\Omega)}, \\ &\leq C\|\nabla u_n\|_{L^2(\Omega)}^2 + C\|\nabla d_n\|_{L^4(\Omega)}^2 + C\|\Delta d_n - f(d_n)\|_{L^2(\Omega)}, \end{aligned} \quad (3.35)$$

where we used embedding inequality; the values of C are variant. Thus, (3.28), (3.29) and (3.34) yield

$$\|\partial_t d_n\|_{L^2((0,T)\times\Omega)} \leq C. \quad (3.36)$$

Summing up to the previous results, by taking a subsequence if necessary, we can assume that:

$$\begin{aligned} d_n &\rightarrow d \text{ in } C([0, T]; L_{weak}^2(\Omega)), \\ d_n &\rightarrow d \text{ weakly in } L^\infty([0, T]; H^1(\Omega)) \cap L^2([0, T]; H^2(\Omega)), \\ d_n &\rightarrow d \text{ strongly in } L^2([0, T]; H^1(\Omega)) \\ \nabla d_n &\rightarrow \nabla d \text{ weakly in } L^4((0, T) \times \Omega), \\ \Delta d_n - f(d_n) &\rightarrow \Delta d - f(d) \text{ weakly in } L^2([0, T]; L^2(\Omega)), \\ F(d_n) &\rightarrow F(d) \text{ strongly in } L^2([0, T]; H^1(\Omega)). \end{aligned}$$

Now, we consider the convergence of the terms relate to d_n and ∇d_n . Let ϕ be a test function, then

$$\begin{aligned} &\int_{\Omega} (\nabla d_n \odot \nabla d_n - \nabla d \odot \nabla d) \cdot \nabla \phi dx dt \\ &\leq \int_{\Omega} (\nabla d_n \odot \nabla d_n - \nabla d_n \odot \nabla d) \cdot \nabla \phi dx dt \\ &+ \int_{\Omega} (\nabla d_n \odot \nabla d - \nabla d \odot \nabla d) \cdot \nabla \phi dx dt \\ &\leq C\|\nabla d_n\|_{L^2(\Omega)}\|\nabla d_n - \nabla d\|_{L^2(\Omega)} \\ &+ C\|\nabla d_n\|_{L^2(\Omega)}\|\nabla d_n - \nabla d\|_{L^2(\Omega)}. \end{aligned}$$

By the strong convergence of ∇d_n in $L^2(\Omega)$, we conclude that

$$\nabla d_n \odot \nabla d_n \rightarrow \nabla d \odot \nabla d \text{ in } D'(\Omega \times (0, T)).$$

Similarly,

$$\frac{1}{2}|\nabla d_n|^2 I_3 \rightarrow \frac{1}{2}|\nabla d|^2 I_3 \text{ in } D'(\Omega \times (0, T)).$$

and

$$u_n \nabla d_n \rightarrow u \nabla d \text{ in } D'(\Omega \times (0, T)).$$

where we have used

$$u_n \rightarrow u \text{ weakly in } L^2([0, T]; H_0^1(\Omega)).$$

Therefore, (3.1d) and (3.9) hold at least in the sense of distribution. Moreover, by the uniform estimates on u, d and (3.1d), we know that the map

$$t \rightarrow \int_{\Omega} d_n(x, t) \phi(x) dx \text{ for any } \phi \in \mathcal{D}(\Omega),$$

is equi-continuous on $[0, T]$. By the Ascoli-Arzelà Theorem, we know that

$$t \rightarrow \int_{\Omega} d_n(x, t) \phi(x) dx,$$

is continuous for any $\phi \in \mathcal{D}(\Omega)$. Thus, d satisfies the initial condition in (3.1d).

Now, multiplying (3.12) by a test function $\phi, \phi \in C^2([0, T] \times \Omega), \phi \geq 0, \phi(T, \cdot) = 0, \nabla \phi \cdot n = 0$ we obtain,

$$\begin{aligned} & \int_0^T \int_{\Omega} (\delta + \rho_n) \mathcal{H}(\theta_n) + \rho_n \mathcal{H}(\theta_n) u_n \cdot \nabla \phi dx dt + \int_0^T \int_{\Omega} \mathcal{K}_h(\theta_n) \Delta \phi - \delta \theta_n^{\alpha+1} h(\theta_n) \phi dx \\ & \leq \int_0^T \int_{\Omega} ((\delta - 1) h(\theta_n) \mathbb{S}_n : \nabla u_n + \kappa(\theta_n) h'(\theta_n) |\nabla \theta_n|^2) \phi dx dt \\ & + \int_0^T \int_{\Omega} R \theta_n h(\theta_n) \rho_n \operatorname{div} u_n dx - \int_{\Omega} h(\theta_n) |\Delta d_n - f(d_n)|^2 dx dt \\ & + \varepsilon \int_0^T \int_{\Omega} \nabla \rho \cdot \nabla ((H(\theta_n) - \theta_n h(\theta_n)) \phi) dx dt - \int_{\Omega} (\rho_{0,\delta} + \delta) H(\theta_{0,\delta}) \phi(0) dx, \end{aligned} \tag{3.37}$$

To conclude we can pass to the limit for $n \rightarrow \infty$ in (3.37) to obtain a renormalized thermal energy inequality:

$$\begin{aligned} & \int_0^T \int_{\Omega} (\delta + \rho) \mathcal{H}(\theta) + \rho \mathcal{H}(\theta) u \cdot \nabla \phi dx dt + \int_0^T \int_{\Omega} \mathcal{K}_h(\theta) \delta \phi - \delta \theta^{\alpha+1} h(\theta) \phi dx \\ & \leq \int_0^T \int_{\Omega} ((\delta - 1) h(\theta) \mathbb{S} : \nabla u + \kappa(\theta) h'(\theta) |\nabla \theta|^2) \phi dx dt \\ & + \int_0^T \int_{\Omega} R \theta h(\theta) \rho \operatorname{div} u dx - \int_{\Omega} h(\theta) |\Delta d - f(d)|^2 dx dt \\ & + \varepsilon \int_0^T \int_{\Omega} \nabla \rho \cdot \nabla ((H(\theta) - \theta h(\theta)) \phi) dx dt - \int_{\Omega} (\rho_{0,\delta} + \delta) H(\theta_{0,\delta}) \phi(0) dx, \end{aligned} \tag{3.38}$$

Finally, multiplying the energy inequality (3.10) by a function $\phi \in C^\infty[0, T]$, $\phi(0) = 1$, $\phi(T) = 0$, $\partial_t \phi \leq 0$, and integrating by parts we infer

$$\begin{aligned} & \int_0^T \int_\Omega (-\partial_t \phi) \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\delta}{\beta-1} \rho^\beta + \frac{|\nabla d|^2}{2} + F(d) + \rho \theta + \delta \theta \right) dx dt \\ & + \delta \int_0^T \int_\Omega \phi (\mathbb{S} : \nabla u + \theta^{\alpha+1}) dx dt \\ & \leq \int_0^T \int_\Omega \frac{1}{2} \frac{|m_0|^2}{\rho_0} + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\delta}{\beta-1} \rho_0^\beta + \frac{|\nabla d_0|^2}{2} + F(d_0) + \rho_0 \theta_0 + \delta \theta_0 dx. \end{aligned} \quad (3.39)$$

Now we have the existence of a global solution to (3.1)-(3.3) as follows:

Proposition 3.2. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain of the class $C^{2+\nu}$, $\nu > 0$; and there exists a constant $C_0 > 0$, such that $d \cdot f(d) \geq 0$ for all $|d| \geq C_0 > 0$. Let $\varepsilon > 0$, $\delta > 0$, and $\beta > \max\{4, \gamma\}$ be fixed. Then for any given $T > 0$, there is a solution (ρ, u, d) to the initial-boundary value problem of (3.1)-(3.3) in the following sense:*

(1) *The density ρ is a nonnegative function such that*

$$\rho \in L^\gamma([0, T]; W^{2,r}(\Omega)), \partial_t \rho, \Delta \rho \in L^\gamma((0, T) \times \Omega),$$

for some $r > 1$, the velocity $u \in L^2([0, T]; H_0^1(\Omega))$, and (3.1a) holds almost everywhere on $(0, T) \times \Omega$, and the initial and boundary data on ρ are satisfied in the sense of traces. Moreover, the total mass is conserved, that is

$$\int_\Omega \rho(x, t) dx = \int_\Omega \rho_{\delta,0} dx,$$

for all $t \in [0, T]$; and the following inequalities hold

$$\begin{aligned} & \int_0^T \int_\Omega \rho^{\beta+1} dx dt \leq C(\varepsilon, \delta), \\ & \varepsilon \int_0^T \int_\Omega |\nabla \rho|^2 dx dt \leq C \quad \text{with } C \text{ independent of } \varepsilon. \end{aligned} \quad (3.40)$$

(2) *The modified momentum equation (3.1b) is satisfied in $\mathcal{D}'((0, T) \times \Omega)$. Moreover,*

$$\rho u \in C([0, T]; L_{weak}^{2\gamma/(\gamma+1)}(\Omega)),$$

satisfied the initial conditions (3.3b).

(3) *The energy inequality (3.39) holds for any function, $\phi \in C^\infty[0, T]$, $\phi(0) = 1$, $\phi(T) = 0$, $\partial_t \phi \leq 0$.*

(4) *All terms in (3.1c) are locally integrable on $(0, T) \times \Omega$. The direction d satisfies the equation (3.1c) and the initial data (3.3c) in the sense of distribution.*

(5) *The temperature θ is a non-negative function,*

$$\theta \in L^{\alpha+1}((0, T) \times \Omega), \theta^{\alpha/2} \in L^2(0, T; W^{1,2}(\Omega)),$$

satisfied the renormalized thermal energy inequality (3.38).

4 Vanishing artificial viscosity

Our next goal is to let the artificial viscosity $\varepsilon \rightarrow 0$ in the approximating system (3.1)-(3.3). Here we denote by $\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon$ the corresponding solution of the approximate problems (3.1)-(3.3) whose existence of which was stated in proposition 3.2, we point out that at this stage the boundedness of $\nabla \rho_\varepsilon$ is no longer hold and, consequently, strong convergence of the sequence $\{\rho_\varepsilon\}_{\varepsilon>0}$ in $L^1((0, T) \times \Omega)$ becomes a central issue.

4.1. Pressure and temperature estimates

In order to avoid concentrations in the pressure term, we have to find a bound in a reflexive space $L^p((0, T) \times \Omega)$, with $p > 1$, independent of ε . Note that the estimate of this type can be deduced via the multiplier of the form

$$\psi \mathcal{B}[\rho_\varepsilon - \frac{1}{\Omega} \rho_\varepsilon dx], \quad \psi \in \mathcal{D}(0, T),$$

in the regularized momentum equation (3.1b). Here, the symbol $\mathcal{B} \approx \operatorname{div}^{-1}$ stands for the so-called Bogovskii operator-a suitable branch of solutions to the problem

$$\operatorname{div} \mathcal{B}[h] = h, \mathcal{B}[h]|_{\partial\Omega} = 0, \int_{\Omega} h dx = 0.$$

Similarly to Section 5 of [1], such a procedure yields an estimate

Lemma 4.1. *There is a constant C such that*

$$\int_0^T \int_{\Omega} (\rho_\varepsilon^\gamma + \rho_\varepsilon \theta_\varepsilon + \delta \rho_\varepsilon^\beta) \rho_\varepsilon dx dt \leq C, \quad (4.1)$$

As far as the temperature is concerned, it is sufficient to set

$$h(\theta) = \frac{1}{(1 + \theta)^\omega}, \quad \omega \in (0, 1) \quad \phi(t, x) = \psi(t), \quad 0 \leq \psi \leq 1, \psi \in \mathcal{D}(0, T).$$

in (3.38). Following the line of argument as in section 2 we get

$$\theta_\varepsilon^{\frac{\alpha+1-\omega}{2}} \text{ bounded in } L^2(0, T; W^{1,2}(\Omega)) \text{ for any } \omega \in (0, 1). \quad (4.2)$$

4.2. The vanishing viscosity limit passage

From the previous estimates, we have

$$\varepsilon \Delta \rho_\varepsilon \rightarrow 0 \text{ in } L^2(0, T; W^{-1,2}(\Omega)) \quad (4.3)$$

and

$$\varepsilon \nabla \rho_\varepsilon \nabla \rho_\varepsilon \rightarrow 0 \text{ in } L^1(0, T; L^1(\Omega)) \quad (4.4)$$

as $\varepsilon \rightarrow 0$. So far, we may assume that

$$\begin{cases} \rho_\varepsilon \rightarrow \rho & \text{in } C(0, T; L_{weak}^\gamma(\Omega)), \\ u_\varepsilon \rightarrow u & \text{weakly in } L^2(0, T; W_0^{1,2}(\Omega)), \\ \rho_\varepsilon u_\varepsilon \rightarrow \rho_\varepsilon u_\varepsilon & \text{weakly in } C(0, T; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega)), \\ \theta_\varepsilon^{\alpha/2} \rightarrow \theta_\varepsilon^{\alpha/2} & \text{weakly} - (\star) \text{ in } L^\infty([0, T]; L^{2/\alpha}(\Omega)) \cap L^2([0, T]; H^1(\Omega)), \\ d_\varepsilon \rightarrow d & \text{weakly in } L^\infty([0, T]; H^1(\Omega)) \cap L^2([0, T]; H^2(\Omega)), \end{cases} \quad (4.5)$$

Similarly to section 3, we still can deduce

$$\begin{cases} \theta_\varepsilon \rightarrow \theta & \text{strongly in } L^2([0, T]; L^2(\Omega)), \\ d_\varepsilon \rightarrow d & \text{strongly in } L^2([0, T]; L^2(\Omega)), \end{cases} \quad (4.6)$$

Consequently, letting $\varepsilon \rightarrow 0$ and making use of (4.1)-(4.6), we conclude that the limit $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, d_\varepsilon)$ satisfies the following system:

$$\begin{cases} \partial_t(\rho) + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \overline{P} = \operatorname{div} \mathbb{S} - \nu \operatorname{div}(\nabla d \odot \nabla d - (\frac{1}{2}|\nabla d|^2 + F(d))\mathbb{I}), \\ \partial_t(\rho \theta) + \operatorname{div}(\rho \theta u) + \operatorname{div} q = \mathbb{S} : \nabla u - R \rho \theta \operatorname{div} u + |\Delta d - f(d)|^2, \\ \partial_t d + u \cdot \nabla d = \Delta d - f(d), \quad |d| = 1, \end{cases} \quad (4.7)$$

where $\overline{P} = \overline{\rho^\gamma + R \rho \theta + \delta \rho^\beta}$, here $\overline{K}(x)$ stands for a weak limit of $\{K_\varepsilon\}$.

4.3. the strong convergence of density

We observe that $\rho_\varepsilon, u_\varepsilon$ is a strong solution of parabolic equation (3.1a), then the renormalized form can be written as

$$\begin{aligned} & \partial_t b(\rho_\varepsilon) + \operatorname{div}(b(\rho_\varepsilon)u_\varepsilon) + (b' \rho_\varepsilon - b(\rho_\varepsilon)) \operatorname{div} u_\varepsilon \\ & = \varepsilon \operatorname{div}(\chi_\Omega \nabla b(\rho_\varepsilon)) - \varepsilon \chi_\Omega b''(\rho_\varepsilon) |\nabla \rho_\varepsilon|^2, \end{aligned} \quad (4.8)$$

in $D'((0, T) \times R^3)$, with $b \in C^2[0, \infty)$, $b(0) = 0$, and b', b'' bounded functions and b convex, where χ_Ω is the characteristics function of Ω . By virtue of (4.8) and the convexity of b , we have

$$\int_0^T \int_\Omega \psi(b' \rho_\varepsilon - b(\rho_\varepsilon)) \operatorname{div} u_\varepsilon dx dt \leq \int_\Omega b(\rho_{0,\delta}) dx + \int_0^T \int_\Omega \partial_t \psi b(\rho_\varepsilon) dx dt,$$

for any $\psi \in C^\infty[0, T]$, $0 \leq \psi \leq 1$, $\psi(0) = 1$, $\psi(T) = 0$. Taking $b(z) = z \log z$ gives us the following estimates:

$$\int_0^T \int_\Omega \psi \rho_\varepsilon \operatorname{div} u_\varepsilon dx dt \leq \int_\Omega \rho_{0,\delta} \log(\rho_{0,\delta}) dx + \int_0^T \int_\Omega \partial_t \psi \rho_\varepsilon \log(\rho_\varepsilon) dx dt,$$

and letting $\varepsilon \rightarrow 0$ yields

$$\int_0^T \int_\Omega \psi \overline{\rho \operatorname{div} u} dx dt \leq \int_\Omega \rho_{0,\delta} \log(\rho_{0,\delta}) dx + \int_0^T \int_\Omega \partial_t \psi \overline{\rho \log(\rho)} dx dt,$$

that is,

$$\int_0^T \int_\Omega \overline{\psi \rho \operatorname{div} u} dx dt \leq \int_\Omega \rho_{0,\delta} \log(\rho_{0,\delta}) dx + \int_0^T \int_\Omega \overline{\rho \log(\rho)} dx dt, \quad (4.9)$$

Meanwhile, (ρ, u) satisfies

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + (b'(\rho) - b(\rho))\operatorname{div}u = 0, \quad (4.10)$$

Using (4.10) and $b(z) = z \log z$, we deduce the following inequality:

$$\int_0^T \int_{\Omega} \psi \rho \operatorname{div}u dx dt \leq \int_{\Omega} \rho_{0,\delta} \log(\rho_{0,\delta}) dx + \int_0^T \int_{\Omega} \rho \log(\rho) dx dt, \quad (4.11)$$

From (4.11) and (4.9), we deduce that

$$\int_{\Omega} (\overline{\rho \log \rho} - \rho \log \rho)(\tau) dx \leq \int_0^T \int_{\Omega} \rho \operatorname{div}u - \overline{\rho \operatorname{div}u} dx dt, \quad (4.12)$$

for all most everywhere $\tau \in [0, T]$.

To obtain the strong convergence of density ρ_ε , the crucial point is to get the weak continuity of the viscous pressure, namely:

Lemma 4.2. *Let $(\rho_\varepsilon, u_\varepsilon)$ be the sequence of approximate solutions constructed in Proposition 4.1, then*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \int_0^T \int_{\Omega} \psi \eta (\rho_\varepsilon^\gamma + \delta \rho^\beta - \operatorname{div}u_\varepsilon) \rho_\varepsilon dx dt \\ &= \int_0^T \int_{\Omega} \psi \eta (\overline{P} - \operatorname{div}u) \rho dx dt \quad \text{for any } \psi \in \mathcal{D}(0, T), \eta \in \mathcal{D}(\Omega), \end{aligned}$$

where $\overline{P} = \overline{\rho^\gamma + R\rho\theta + \delta\rho^\beta}$.

The detail of this proof can be seen in [1].

From Lemma 4.2, we have

$$\int_0^T \int_{\Omega} \rho \operatorname{div}u - \overline{\rho \operatorname{div}u} dx dt \leq \frac{1}{\mu} \int_0^T \int_{\Omega} (\overline{P}\rho - \overline{\rho^\gamma + \delta\rho^{\beta+1}}), \quad (4.13)$$

By (4.12) and (4.13), we conclude that

$$\int_{\Omega} \overline{\rho \log \rho} - \rho \log \rho dx \leq \frac{1}{\mu} \int_0^T \int_{\Omega} (\overline{P}\rho - \overline{\rho^\gamma + \delta\rho^{\beta+1}}),$$

and

$$\overline{P}\rho - \overline{\rho^\gamma + \delta\rho^{\beta+1}} \leq 0,$$

Due to the convexity of $\rho^\gamma + \delta\rho^\beta$. So

$$\int_{\Omega} \overline{\rho \log \rho} - \rho \log \rho dx \leq 0,$$

On the other hand,

$$\overline{\rho \log \rho} - \rho \log \rho \geq 0,$$

Consequently $\overline{\rho \log \rho} = \rho \log \rho$ that means

$$\rho_\varepsilon \rightarrow \rho, \quad \text{in } L^1((0, T) \times \Omega).$$

Thus, we can pass to the limit as $\varepsilon \rightarrow 0$ to obtain the following result:

Proposition 4.3. Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain of the class $C^{2+\nu}$, $\nu > 0$; and there exists a constant $C_0 > 0$, such that $d \cdot f(d) \geq 0$ for all $|d| \geq C_0 > 0$. Let $\varepsilon > 0, \delta > 0$, and $\beta > \max\{4, \gamma\}$ be fixed. Then for any given $T > 0$, there is a solution (ρ, u, d) to the initial-boundary value problem of (3.1)-(3.3) in the following sense:

(1) The density ρ is a nonnegative function such that

$$\rho \in C([0, T]; L_{weak}^\beta(\Omega)), \rho \in L^{\beta+1}((0, T) \times \Omega),$$

satisfying the initial condition (3.3a). Moreover the velocity $u \in L^2([0, T]; H_0^1(\Omega))$, and ρ, u , solves the continuity equation in (1.1) in $D'((0, T) \times \mathbb{R}^3)$ provided they were extended to be zero outside Ω .

(2) The functions ρ, u, θ, d satisfy the system (4.7) in $\mathcal{D}'((0, T) \times \Omega)$. Moreover,

$$\rho u \in C([0, T]; L_{weak}^{2\gamma/(\gamma+1)}(\Omega)),$$

(3) The energy inequality

$$\begin{aligned} & \int_0^T \int_\Omega (-\partial_t \phi) \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\delta}{\beta-1} \rho^\beta + \frac{|\nabla d|^2}{2} + F(d) + \rho \theta + \delta \theta \right) dx dt \\ & + \delta \int_0^T \int_\Omega \phi (\mathbb{S} : \nabla u + \theta^{\alpha+1}) dx dt \\ & \leq \int_0^T \int_\Omega \frac{1}{2} \frac{|m_0|^2}{\rho_0} + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\delta}{\beta-1} \rho_0^\beta + \frac{|\nabla d_0|^2}{2} + F(d_0) + \rho_0 \theta_0 + \delta \theta_0 dx. \end{aligned} \quad (4.14)$$

holds for any function $\phi \in C^\infty[0, T]$, $\phi(0) = 1, \phi(T) = 0, \partial_t \phi \leq 0$.

(5) The temperature θ is a non-negative function,

$$\theta \in L^{\alpha+1}((0, T) \times \Omega), \theta^{\frac{\alpha-\omega+1}{2}} \in L^2(0, T; W^{1,2}(\Omega)),$$

satisfied the renormalized thermal energy inequality

$$\begin{aligned} & \int_0^T \int_\Omega (\delta + \rho) \mathcal{H}(\theta) + \rho \mathcal{H}(\theta) u \cdot \nabla \phi dx dt + \int_0^T \int_\Omega \mathcal{K}_h(\theta) \delta \phi - \delta \theta^{\alpha+1} h(\theta) \phi dx \\ & \leq \int_0^T \int_\Omega ((\delta - 1) h(\theta) \mathbb{S} : \nabla u + \kappa(\theta) h'(\theta) |\nabla \theta|^2) \phi dx dt \\ & + \int_0^T \int_\Omega R \theta h(\theta) \rho \operatorname{div} u dx - \int_\Omega h(\theta) |\Delta d - f(d)|^2 dx dt \\ & + \varepsilon \int_0^T \int_\Omega \nabla \rho \cdot \nabla ((H(\theta) - \theta h(\theta)) \phi) dx dt - \int_\Omega (\rho_{0,\delta} + \delta) H(\theta_{0,\delta}) \phi(0) dx, \end{aligned} \quad (4.15)$$

for any function $\phi \in C^\infty[0, T]$, $\phi(T) = 0$.

5 Vanishing artificial pressure limit

The objective of this section is to recover the original system by vanishing the parameter δ . Denote $\rho_\delta, u_\delta, \theta_\delta, d_\delta$ the corresponding approximate solutions constructed in Proposition 4.3. Again, in this part the crucial issue is to recover the strong convergence for ρ_δ in L^1 space.

5.1. Uniform estimates

To begin with, the energy inequality can be used to deduce the estimates

$$\rho_\delta \text{ bounded in } L^\infty(0, T; L^\gamma(\Omega)), \quad (5.1)$$

$$\sqrt{\rho_\delta} u_\delta \text{ bounded in } L^\infty(0, T; L^2(\Omega)), \quad (5.2)$$

$$(\delta + \rho_\delta) \theta_\delta \text{ bounded in } L^\infty(0, T; L^1(\Omega)), \quad (5.3)$$

and

$$\delta \int_0^T \int_\Omega \theta_\delta^{\alpha+1} dx dt \leq C, \quad (5.4)$$

Now take

$$\varphi(t, x) = \psi(t), \quad 0 \leq \psi \leq 1, \psi \in \mathcal{D}(0, T), h(\theta) = \frac{\omega}{\omega + \theta}, \omega > 0, \quad (5.5)$$

in (4.15) to deduce

$$\begin{aligned} & \int_0^T \int_\Omega \left(\frac{1-\delta}{\omega + \theta_\delta} \mathbb{S}_\delta : \nabla u + \frac{\kappa(\theta_\delta)}{(\omega + \theta_\delta)^2} |\nabla \theta_\delta|^2 + \frac{1}{\omega + \theta_\delta} |\Delta d_\delta - f(d_\delta)|^2 \right) \psi dx dt \\ & \leq \int_0^T \int_\Omega \frac{\theta_\delta}{\omega + \theta_\delta} \rho_\delta \operatorname{div} u_\delta dx dt + \delta \int_0^T \int_\Omega \theta^\alpha dx dt \\ & \quad - \int_\Omega (\rho_{0,\delta} + \delta) \mathcal{H}_{h,\omega}(\theta_{0,\delta}) dx + \int_\Omega (\rho_\delta + \delta) \mathcal{H}_{h,\omega}(\theta_\delta) dx, \end{aligned} \quad (5.6)$$

where

$$\mathcal{H}_{h,\omega}(\theta) = \int_1^\theta \frac{1}{\omega + z} dz.$$

Utilizing the estimates already obtained, we take the limit for $\omega \rightarrow 0$ to get

$$\begin{aligned} & \int_0^T \int_\Omega \left(\frac{1-\delta}{\theta_\delta} \mathbb{S}_\delta : \nabla u_\delta + \frac{\kappa(\theta_\delta)}{\theta_\delta^2} |\nabla \theta_\delta|^2 + \frac{1}{\theta_\delta} |\Delta d_\delta - f(d_\delta)|^2 \right) dx dt \\ & \leq C \int_0^T \int_\Omega \rho_\delta \operatorname{div} u_\delta dx dt + C \end{aligned} \quad (5.7)$$

with C independent of $\delta > 0$. The right-hand side of above inequality is bounded. Thus, utilizing above inequality together with hypothesis (3.2), we have

$$\nabla \log \theta_\delta \text{ bounded in } L^2(0, T; L^2(\Omega)), \quad (5.8)$$

and

$$\nabla \theta_\delta^{\alpha/2} \text{ bounded in } L^2(0, T; L^2(\Omega)), \quad (5.9)$$

Following the line of argument in previous, we continue to deduce

$$\left\{ \begin{array}{l} \rho_\delta \rightarrow \rho \text{ in } C(0, T; L_{weak}^\gamma(\Omega)), \\ u_\delta \rightarrow u \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega)), \\ \rho_\delta u_\delta \rightarrow \rho u \text{ weakly in } C(0, T; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega)), \\ \theta_\delta^{\alpha/2} \rightarrow \theta^{\alpha/2} \text{ weakly} - (\star) \text{ in } L^\infty([0, T]; L^{2/\alpha}(\Omega)) \cap L^2([0, T]; H^1(\Omega)), \\ d_\delta \rightarrow d \text{ weakly in } L^\infty([0, T]; H^1(\Omega)) \cap L^2([0, T]; H^2(\Omega)), \\ \theta_\delta \rightarrow \theta \text{ strongly in } L^2([0, T]; L^2(\Omega)), \\ d_\delta \rightarrow d \text{ strongly in } L^2([0, T]; L^2(\Omega)), \end{array} \right. \quad (5.10)$$

5.2. Refined temperature and pressure estimates

Following the arguments of Chapter 7 in [4], we can derive the estimate of θ_δ in the space $L^{\alpha+1}((0, T) \times \Omega)$. The main idea is the same as in [4], that means, we use the quantity

$$\psi(t, x) = \phi(t)(\eta - \underline{\eta}), \quad 0 \leq \phi \leq 1, \phi \in \mathcal{D}(0, T),$$

where η is the following of the Neumann problem

$$\begin{aligned} \Delta \eta &= b(\rho_\delta) - \frac{1}{|\Omega|} b(\rho_\delta) dx \quad \text{in } \Omega, \\ \nabla \eta \cdot n|_{\partial \Omega} &= 0, \quad \int_{\Omega} \eta dx = 0. \end{aligned}$$

with a suitable chosen function b , as a test function in the thermal energy inequality (4.15). Following step by step the proof in Section 7.5.2 of [4], we infer that

$$\theta_\delta \text{ is bounded in } L^{\alpha+1}((0, T) \times \Omega). \quad (5.11)$$

Next, pursuing the approach of previous section, we deduce the estimate

$$\int_0^T \int_{\Omega} (\rho_\delta^\gamma + R\rho_\delta \theta_\delta + \delta \rho_\delta^\beta) \rho_\delta dx dt \leq C; . \quad (5.12)$$

5.3. The Limit Passage

By the energy estimate, we get

$$\delta \rho_\delta^\beta \rightarrow 0 \text{ in } L^1((0, T) \times \Omega) \text{ as } \delta \rightarrow 0, \quad (5.13)$$

Consequently, Letting $\delta \rightarrow 0$ in (4.7) and making use of (5.1)-(5.13), the limit of $(\rho_\delta, u_\delta, \theta_\delta, d_\delta)$ satisfies the following system:

$$\begin{cases} \partial_t(\rho) + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \overline{P} = \operatorname{div} \mathbb{S} - \nu \operatorname{div}(\nabla d \odot \nabla d - (\frac{1}{2} |\nabla d|^2 + F(d)) \mathbb{I}), \\ \partial_t d + u \cdot \nabla d = \Delta d - f(d), \quad |d| = 1, \\ \partial_t(\rho \theta) + \operatorname{div}(\rho \theta u) + \operatorname{div} q = \mathbb{S} : \nabla u - R\rho \theta \operatorname{div} u + |\Delta d - f(d)|^2, \end{cases} \quad (5.14)$$

where $\overline{P} = \overline{\rho^\gamma + R\rho\theta}$.

5.4. the strong convergence of density

In order to complete the proof of Theorem 1.2, we will need to show the strong convergence of ρ_δ in $L^1(\Omega)$, or, equivalently $\overline{\rho^\gamma + R\rho\theta} = \rho^\gamma + R\rho\theta$.

Since ρ_δ, u_δ is a renormalized solution of the continuity equation (5.14) in $\mathcal{D}'((0, T) \times R^3)$, we have

$$T_k(\rho_\delta)_t + \operatorname{div}(T_k(\rho_\delta)u_\delta) + (T'_k(\rho_\delta)\rho_\delta - T_k(\rho_\delta))\operatorname{div}(u_\delta) = 0 \text{ in } \mathcal{D}'((0, T) \times R^3),$$

where $T_k(z) = kT(\frac{z}{k})$ for $z \in R$, $k=1,2,3\dots$ and $T \in C^\infty(R)$ is chosen so that

$$T(z) = z \text{ for } z \leq 1, \quad T(z) = 2 \text{ for } z \geq 3, \quad T \text{ convex.}$$

Passing to the limit for $\delta \rightarrow 0$ we deduce that

$$\overline{T_k(\rho)}_t + \operatorname{div}(\overline{T_k(\rho)u}) + \overline{(T'_k(\rho)\rho - T_k(\rho))\operatorname{div}u} = 0 \quad \text{in } \mathcal{D}'((0,T) \times R^3),$$

where

$$(T'_k(\rho_\delta)\rho_\delta - T_k(\rho_\delta))\operatorname{div}(u_\delta) \rightarrow \overline{(T'_k(\rho)\rho - T_k(\rho))\operatorname{div}u} \quad \text{weakly in } L^2((0,T) \times \Omega),$$

and

$$T_k(\rho_\delta) \rightarrow \overline{T_k(\rho)} \quad \text{in } C([0,T]; L^p_{weak}(\Omega)), \quad \text{for all } 1 \leq p < \infty.$$

using the function

$$\varphi(t, x) = \psi\eta(x)A_i[T_k(\rho_\delta)], \quad \psi \in \mathcal{D}[0,T], \eta \in (\Omega),$$

as a test function for momentum equation in (4.7), by a similar calculation to the previous sections, we can deduce the following result:

Lemma 5.1. *Let (ρ_δ, u_δ) be the sequence of approximate solutions constructed in Proposition 4.3, then*

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_0^T \int_\Omega \psi\eta(\rho_\delta^\gamma - \operatorname{div}u_\delta)\rho_\delta dxdt \\ &= \int_0^T \int_\Omega \psi\eta(\overline{P} - \operatorname{div}u)\rho dxdt \quad \text{for any } \psi \in \mathcal{D}(0,T), \eta \in \mathcal{D}(\Omega), \end{aligned}$$

where $\overline{P} = \overline{\rho^\gamma + R\rho\theta}$.

In order to get the strong convergence of ρ_δ , we need to define the oscillation defect measure as follows:

Lemma 5.2. *There exists a constant C independent of k such that*

$$OSC_{\gamma+1}[\rho_\delta \rightarrow \rho]((0,T) \times \Omega) \leq C$$

for any $k \geq 1$.

We are now ready to show the strong convergence of the density. To this end, we introduce a sequence of functions $L_k \in C^1(R)$:

$$L_k(z) = \begin{cases} z \ln z, & 0 \leq z < k \\ z \ln(k) + z \int_k^z \frac{T_k(s)}{s^2} ds, & z \geq k. \end{cases}$$

Noting that L_k can be written as

$$L_k(z) = \beta_k z + b_k z,$$

We deduce that

$$\partial_t L_k(\rho_\delta) + \operatorname{div}(L_k(\rho_\delta)u_\delta) + T_k(\rho_\delta)\operatorname{div}u_\delta = 0, \quad (5.15)$$

and

$$\partial_t L_k(\rho) + \operatorname{div}(L_k(\rho)u) + T_k(\rho)\operatorname{div}u = 0, \quad (5.16)$$

in $\mathcal{D}'((0, T) \times \Omega)$. Letting $\delta \rightarrow 0$, we can assume that

$$L_k(\rho_\delta) \rightarrow \overline{L_k(\rho)} \text{ in } C([0, T]; L_{weak}^\gamma(\Omega)).$$

Taking the difference of (5.15) and (5.16), and integrating with respect to time t, we obtain

$$\begin{aligned} & \int_{\Omega} (L_k(\rho_\delta) - L_k(\rho))\phi dx \\ &= \int_0^T \int_{\Omega} (L_k(\rho_\delta)u_\delta - L_k(\rho)u) \cdot \nabla \phi + (T_k(\rho)\operatorname{div}u - T_k(\rho_\delta)\operatorname{div}u_\delta\phi) dx dt, \end{aligned} \quad (5.17)$$

for any $\phi \in \mathcal{D}(\Omega)$. Following the line of argument in [1], we get

$$\begin{aligned} & \int_{\Omega} (\overline{L_k(\rho)} - L_k(\rho))\phi dx \\ &= \int_0^T \int_{\Omega} T_k(\rho)\operatorname{div}u dx dt - \lim_{\delta \rightarrow 0^+} \int_0^T \int_{\Omega} T_k(\rho_\delta)\operatorname{div}u_\delta\phi dx dt, \end{aligned} \quad (5.18)$$

We observe that the term $\overline{L_k(\rho)} - L_k(\rho)$ is bounded by its definition. Using Lemmas 5.2 and the monotonicity of the pressure, we can estimate the right-hand side of (5.18)

$$\begin{aligned} & \int_0^T \int_{\Omega} T_k(\rho)\operatorname{div}u dx dt - \lim_{\delta \rightarrow 0^+} \int_0^T \int_{\Omega} T_k(\rho_\delta)\operatorname{div}u_\delta\phi dx dt \\ & \leq \int_0^T \int_{\Omega} (\overline{L_k(\rho)} - L_k(\rho))\operatorname{div}u dx dt, \end{aligned} \quad (5.19)$$

By virtue of Lemma 5.2, the right-hand side of (5.19) tends to zero as $k \rightarrow \infty$. So we conclude that

$$\overline{\rho \log \rho(t)} = \rho \log \rho(t),$$

as $k \rightarrow \infty$. Thus we obtain the strong convergence of ρ_δ in $L^1((0, T) \times \Omega)$.

Therefore we complete the proof of Theorem 1.2.

Acknowledgements

This research was supported in part by NNSFC(Grant No.11271381) and China 973 Program(Grant No. 2011CB808002).

References

- [1] Dehua Wang, Cheng Yu Global weak solution and large-time behavior for the compressible flow of liquid crystals. Arch. Rational Mech. Anal. 204 (2012) 881-915.
- [2] E.Feireisl, M.Fremond, E.Rocca, G.Schimperna, A new approach to non-isothermal models for nematic liquid crystals. Arch. Rational Mech. Anal. 205 (2012) 651-672.

- [3] E. Feireisl, M. Fremond, E. Rocca, G. Schimperna, On a non-isothermal model for nematic liquid crystals. *Nonlinearity* 24, 243C257 (2011)
- [4] E. Firesel, Dynamics of viscous compressible fluids, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004.
- [5] J. K. Li, Z. Xin, Global weak solutions to non-isothermal nematic liquid crystals in 2D. arXiv: 1307.2065
- [6] J. Ericksen, Conservation laws for liquid crystals, *Trans. Soc. Rheol.* 5(1961), 22C34.
- [7] G. Leslie, : Some constitutive equations for liquid crystals. *Arch. Rational Mech. Anal.* 28, 265C283 (1963)
- [8] F. H. Lin, Nonlinear theory of defects in nematic liquid crystals: Phase transition and flow phenomena, *Comm. Pure. Appl. Math.*, 42(1989), 789C814.
- [9] F. H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals. *Commun. Pure Appl. Math.* 48, 501C537 (1995)
- [10] F. H. Lin, C. Liu, Partial regularity of the dynamic system modeling the flow of liquid crystals, *Disc. Cont. Dyn. Sys.* 2(1996), 1C22.
- [11] A. Novotný, I. Straškraba, Introduction to the Mathematical Theory of Compressible Flow, Oxford Univ. Press, Oxford, 2004.
- [12] P. L. Lions, Mathematical topics in fluid dynamics, Vol. 2, Compressible models. Oxford Science Publication, Oxford, 1998.
- [13] F. Jiang and Z. Tan, Global weak solution to the flow of liquid crystals system, *Math. Meth. Appl. Sci.*, 32(2009), 2243C2266.
- [14] X. G. Liu and Z. Y. Zhang, Existence of the flow of liquid crystal system, *Chin. Ann. Math.*, 30A(2009), 1C20.